A geometric insight into the Fokker-Planck equation

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Abstract. *The introduction of an extended evolution space allows to give an intrinsic formulation of the Fokker-Planck equation and makes clear its structural invariance under the chronoprojective geometry. Symmetry algebras and solutions in time dependent parabolic external fields are deduced.*

INTRODUCTION

The theory of infinitesimal symmetries of differential equations originates in Sophus Lie's works based on the infinitesimal properties of continuous groups of transformations. As explained by Elie Cartan, transformations groups are automorphisms groups of particular structures, at present named Cartan structures, at present named Cartan structures, of which the better known are the affine, projective and conformal geometries.

But it is less known that the automorphisms group of a geometric structure can play the role of symmetry group for a differential system if and only if the concerned differential operators possess a reproducing property with respect to the equivalence relation inherent in the geometry, this defines the so-called structural invariance of the differential operators. Now we can ask why this aspect has not been more developed. An element of answer to this question can be found into the fact that very few geometries exist and among them only the

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conformal one deals with physical systems, with the severe restriction that it concerns massless systems only.

Recently we constructed a new geometry for the Newtonian space-time, synthesizing the conformal equivalence notion of two Galilean structures and the projective equivalence notion of two affine connections, and we have shown that it is perfectly convenient for the description of the Newtonian gravitation [1]. It has been called the chronoprojective geometry since the subjacent characteristic homogeneous space is a generalized Möbius space fibered over the projective line which plays the role of time axis.

However difficulties arise when we want to describe mechanical systems with mass. Difficulties of the same kind also appear in group representation in the framework of which they have been solved by introducing projective representations or equivalently true representations of the Bargmann central extension. This last trick reveals also useful from the geometrical point of view. Indeed over a one-dimensional supplemented space-time a new formulation of the chronoprojective geometry can be given which can be presented as a reduction of the conformal one [2], and which is proving to be very elegant and fruitful for the description of classical and quantum systems [3], [4]. So we have good reasons to think that any evolution equation can be put into an intrinsic purely covariant form in this extended geometric framework. To support this thesis, the case of the Fokker-Planck equation is treated in the present paper.

Concerning the supplementary dimension we have found no reason to look for giving it a physical meaning. Its introduction justifies itself from a formal point of view for, as A. Einstein said: «time and space are modes by which we think and not conditions in which we live».

I.A - THE GEOMETRICAL FRAMEWORK

Let $(V_{n+1,1}; \tilde{g})$ be a $(n+2)$-dimensional Lorentzian manifold (i.e. a pseudo-Riemannian manifold with signature n), which is a principal $(\mathbb{R}, +)$-bundle $\pi : (V_{n+1,1}) \to V_{n+1}$ over a $(n+1)$-dimensional connected smooth manifold V_{n+1}. Let ξ be the generator of the additive group $(\mathbb{R}, +)$, ξ is supposed to be null i.e. $\tilde{g}(\xi, \xi) = o$ (1) and covariantly constant with respect to the Levi-Cività connection $\nabla_{\tilde{g}}$ associated to $(V_{n+1,1}; \tilde{g})$ i.e. $\nabla_{\tilde{g}} \xi = o$. Consequently:

i) $\tilde{g}(\xi)$ is a nowhere vanishing closed basic one-form, so we can set $\tilde{g}(\xi) = \pi^* \psi$, where ψ is a closed one-form on V_{n+1}.

ii) ξ is a Killing vector field i.e. $L_\xi \tilde{g} = o$, so, the push-forward twice

(1) In the «relativistic» case $\tilde{g}(\xi, \xi) = c^{-2}$.

contravariant tensor $\gamma \equiv \pi_* \tilde{g}^{-1}$ is well defined on V_{n+1} and satisfies $\gamma(\psi) = o$.

The triplet (V_{n+1}, γ, ψ) can be called a generalized (2) Galilean manifold since a temporal axis is defined by the quotient $V_{n+1}/\mathrm{Ker}\ \psi$ which is supposed to exist.

Let $T : V_{n+1} \to V_{n+1}/\mathrm{Ker}\ \psi$ denote the canonical submersion map. Then each slice $T^{-1}(t)$, $t \in V_{n+1}/\mathrm{Ker}\ \psi$, is endowed with a contravariant «spatial» metric induced by γ and is diffeomorphic to a n-dimensional Riemannian manifold $(V_n; g)$.

As a principal $(\mathbb{R}, +)$-bundle over V_{n+1}, $V_{n+1,1}$ is trivializable, i.e. it possesses a global cross section $\Sigma : V_{n+1} \to V_{n+1,1}$ and correspondingly it is isomorphic to the trivial principal bundle $i : V_{n+1,1} \to V_{n+1} \times (\mathbb{R}, +)$. Let α be the canonical flat connection in $V_{n+1} \times (\mathbb{R}, +)$. If ξ has been chosen such that $(i_* \xi) \lrcorner \alpha = 1$, then:

i) $\tilde{U}_{(i)} = \tilde{g}^{-1}(i^* \alpha)$ is a vector field on $V_{n+1,1}$ and $\pi_* \tilde{U}_{(i)}$ is a vector field on V_{n+1} such that $(\pi_* \tilde{U}_{(i)}) \lrcorner \psi = \xi \lrcorner i^* \alpha = 1$. So that $\tilde{U}_{(i)} = \pi_* U_{(i)}$ plays the role of a Galilean observer.

ii) $\tilde{U}_{(i)} \lrcorner i^* \alpha = \tilde{g}^{-1} (i^* \alpha, i^* \alpha) = 2 \tilde{V}_{(i)}$ is a function defined on the basis since ξ is a Killing vector field and $L_\xi \alpha = o$, so we can set $\tilde{V} = \pi^*(V)$.

Now, some comments about connections are in order. The connection form of the Levi-Civitá connection in $(V_{n+1,1}; \tilde{g})$ takes its values into $o\ (n+1, 1)$ the Lie algebra of the structural group of the bundle $O(V_{n+1,1})$ of pseudo--orthogonal frames over $V_{n+1,1}$. But due to the presence of the null vector field ξ one is led to consider the stabilizer H into $O(n+1, 1)$ of a null direction of $\mathbb{R}^{n+1,1}$. In fact H is isomorphic to the homogeneous Galilei group $H \simeq \mathbb{R}^n \times O(n)$. This then leads to restrict the bundle $O(V_{n+1,1})$ to the bundle $H(V_{n+1,1})$ of Galilean frames over $V_{n+1,1}$. Then the Levi-Civitá connection in $O(V_{n+1,1})$ is reducible to a connection Φ in $H(V_{n+1,1})$ and we shall always denote by $\nabla_{\tilde{g}}$ the corresponding covariant derivative.

Now let us consider the pull-back $(H(V_{n+1}), \Sigma^*\Phi)$ of $(H(V_{n+1,1}), \Phi)$ over V_{n+1} induced by the above considered cross section Σ of V_{n+1} into $V_{n+1,1}$. Since $H(V_{n+1})$ is the bundle of Galilean frames over V_{n+1}, the induced connection $\Sigma^*\Phi$ is a Newtonian connection due to the structure equations of Φ [3]. As such γ and ψ are parallel i.e. such that $\nabla \gamma = o$ and $\nabla \psi = o$ where ∇ denotes the covariant derivative corresponding to $\Sigma^*\Phi$ and the observer $U_{(i)}$ is geodesic i.e. $\nabla_U U = o$.

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(2) The word generalized is used here to remind that one considers a n-dimensional configuration space rather than the usual 3-dimensional space of the Newtonian space-time. But in the following the word generalized will be omitted.

To give a more precise idea of the above construction, let us introduce the adapted coordinate system

$$\{x^\mu\}_{\mu \in [1, n+2]} = \{x^\alpha, x^{n+2}\}_{\alpha \in [1, n+1]}$$

induced by a local chart $\{x^\alpha\}_{\alpha \in [1, n+1]}$ of V_{n+1}.

By setting

(I.1) $$\gamma = \gamma^{\alpha\beta} \partial_\alpha \otimes \partial_\beta$$

(I.2) $$\psi = \psi_\alpha \mathrm{d}x^\alpha$$

from the above given definitions we obtain:

(I.3) $$\tilde{g}^{\alpha\beta} = \gamma^{\alpha\beta}, \qquad \tilde{g}_{\alpha\beta} = \overset{U}{\gamma}_{\alpha\beta} + 2V\psi_\alpha \psi_\beta$$

(I.4) $$\tilde{g}^{\alpha n+2} = U^\alpha, \qquad \tilde{g}_{\alpha n+2} = \psi_\alpha$$

(I.5) $$\tilde{g}^{n+2\, n+2} = 2V, \qquad \tilde{g}_{n+2\, n+2} = 0$$

with

(I.6) $$\psi_\alpha U^\alpha = 1, \qquad \overset{U}{\gamma}_{\alpha\beta} U^\beta = 0, \qquad \overset{U}{\gamma}_{\alpha\tau} \gamma^{\tau\beta} = \delta^\beta_\alpha - U^\beta \psi_\alpha .$$

Each component $\tilde{g}_{\mu\nu}$ or $\tilde{g}^{\mu\nu}$ depends on x^α only and $\xi = \partial_{n+2}$.

The Christoffel's symbols $\tilde{\Gamma}$ of the connection Φ and the ones Γ of $\Sigma^*\Phi$ are given by:

(I.7) $$\tilde{\Gamma}^\gamma_{\alpha\beta} = \overset{U}{\Gamma}{}^\gamma_{\alpha\beta} + \gamma^{\alpha\gamma} \partial_\alpha V \psi_\alpha \psi_\beta$$

(I.8) $$\tilde{\Gamma}^{n+2}_{\alpha\beta} = - \overset{U}{\gamma}_{\tau(\alpha} \nabla_{\beta)} U^\tau - \partial_{(\alpha} V \psi_{\beta)}$$

with

(I.9) $$\overset{U}{\Gamma}{}^\gamma_{\alpha\beta} = \gamma^{\gamma\rho} \left\{ \partial_{(\alpha} \overset{U}{\gamma}_{\beta)\rho} - \frac{1}{2} \partial_\rho \overset{U}{\gamma}_{\alpha\beta} \right\} + U^\gamma \partial_{(\alpha} \psi_{\beta)}$$

and

(I.10) $$\Gamma^\gamma_{\alpha\beta} = \tilde{\Gamma}^\gamma_{\alpha\beta} .$$

We thus see that (I.10) expresses the fact that the connection $\nabla_{\tilde{g}}$ projects onto V_{n+1} as a well defined Newtonian connection ∇.

The non-vanishing components of the Riemann-Christoffel tensor of Φ and $\Sigma^*\Phi$ are expressed by

(I.11) $$\tilde{R}^\delta_{\alpha\beta\gamma} = R^\delta_{\alpha\beta\gamma}$$

(I.12) $$\tilde{R}^{n+2}_{\alpha\beta\gamma} = 2 \nabla_{[\alpha} \tilde{\Gamma}^{n+2}_{\beta]\gamma}$$

$R^\delta_{\alpha\beta\gamma}$ satisfying:

(I.13) $$R^{\delta}_{\alpha\beta\tau}\gamma^{\tau\nu} = R^{\nu}_{\beta\alpha\tau}\gamma^{\tau\delta}$$

which reflects the fact that the subjacent connection $\Sigma^*\Phi$ on V_{n+1} is Newtonian.

I.B - AN INTRINSIC FORMULATION OF THE FOKKER-PLANCK EQUATION

The time dependent probability distribution P of a diffusion process defined on a n-dimensional Riemannian manifold $(V_n; g)$, the state space or configuration space, is governed by the Fokker-Planck equation

(I.14) $$\partial_t P = -\operatorname{div}_g\left(bP - \frac{1}{2}\operatorname{grad} P\right)$$

where b denotes the forward drift, a vector field on V_n.

Hence the Fokker-Planck equation is an evolution equation for the distribution function P on ($V_n \times$ time), the evolution space.

It is well known that the kinematical and dynamical concepts of classical and quantum mechanics are intrinsically formulated over a Galilean manifold, or still better, on the so-called extended evolution space $(V_{n+1,1}; \tilde{g}; \xi)$. Let us then consider the following differential system:

(I.15) $$\operatorname{div}_{\tilde{g}}\left(\tilde{g}^{-1}\left(\left(\tilde{b} - \frac{1}{2}\nabla_{\tilde{g}}\right)\tilde{P}\right)\right) = 0$$

(I.16) $$\xi(\tilde{P}) = 0$$

where

i) $\tilde{b}$ denotes the drift considered here as a closed one-form ($d\tilde{b} = o$) over $V_{n+1,1}$ which satisfies

(I.17) $$\xi \lrcorner \tilde{b} = 1$$

i.e. $\tilde{b}$ is semi-basic for the fibration $\pi : V_{n+1,1} \to V_{n+1}$, and since ξ is a Killing vector field

(I.18) $$[\xi, \tilde{g}^{-1}(\tilde{b})] = 0$$

so that the push-forward vector field $\tilde{g}^{-1}(\tilde{b})$ is well-defined over V_{n+1}: it is the physical drift b which stands in (I.14)

(I.19) $$b = \pi_* \tilde{g}^{-1}(\tilde{b})$$

ii) $\tilde{P}$ is a real function over $V_{n+1,1}$, which is basic from (I.16). We can then set

(I.20) $$\tilde{P} = \pi^* P$$

where P denotes a real function over V_{n+1}.

Note that (I.15) can be written as:

$$\tilde{\nabla}_{\tilde{g}}(\tilde{g}^{-1}(\tilde{b}\tilde{P})) = \frac{1}{2}\,\Delta_{\tilde{g}}\tilde{P} \tag{I.21}$$

where $\Delta_{\tilde{g}}$ denotes the Laplace-Beltrami operator of $(V_{n+1,1};\tilde{g})$.

Now, let us explicitly work out the local expression of this equation in the adapted coordinate system introduced in sec. I.A. One gets:

$$\begin{aligned} &g^{\alpha\beta}\tilde{\nabla}_\alpha(\tilde{P}\tilde{b}_\beta) + g^{\alpha n+2}\{\tilde{\nabla}_\alpha(\tilde{P}\tilde{b}_{n+2}) + \tilde{\nabla}_{n+2}(\tilde{P}\tilde{b}_\alpha)\} + \tilde{g}^{n+2\ n+2}\tilde{\nabla}_{n+2}(\tilde{P}\tilde{b}_{n+2}) = \\ &= \frac{1}{2}\,\{g^{\alpha\beta}\tilde{\nabla}_\alpha\partial_\beta\tilde{P} + g^{\alpha n+2}(\tilde{\nabla}_\alpha\partial_{n+2}\tilde{P} + \tilde{\nabla}_{n+2}\partial_\alpha\tilde{P}) + g^{n+2\ n+2}\tilde{\nabla}_{n+2}\partial_{n+2}\tilde{P}\}. \end{aligned} \tag{I.22}$$

In view of (I.3) - (I.5), (I.7), (I.8) and (I.20), one obtains

$$\begin{aligned} &\gamma^{\alpha\beta}(\nabla_\alpha(P\tilde{b}_\beta) - \tilde{\Gamma}^{n+2}_{\alpha\beta}P\tilde{b}_{n+2}) + U^\alpha(\nabla_\alpha(P\tilde{b}_{n+2}) + P\partial_{n+2}\tilde{b}_\alpha) + 2\,V\partial_{n+2}(P\tilde{b}_{n+2}) = \\ &= \frac{1}{2}\,\{\gamma^{\alpha\beta}(\nabla_\alpha\partial_\beta P - \tilde{\Gamma}^{n+2}_{\alpha\beta}\partial_{n+2}\tilde{P}) + 2\,U^\alpha\partial_\alpha\partial_{n+2}P + 2\,V\partial_{n+2}(\partial_{n+2}P)\}. \end{aligned} \tag{I.23}$$

Now, by taking into account for

– the properties of $\tilde{b}$, namely closure and semi-basicity which imply:

$$\tilde{b}_{n+2} = 1 \tag{I.24}$$

$$\partial_{n+2}\tilde{b}_\alpha = 0 \tag{I.25}$$

– the basicity of $\tilde{P}$ which implies

$$\partial_{n+2}\tilde{P} = 0 \tag{I.26}$$

– the explicit expression (I.8) of $\tilde{\Gamma}^{n+2}_{\alpha\beta}$

one is left with

$$\gamma^{\alpha\beta}\nabla_\alpha(P\tilde{b}_\beta) + (\nabla_\alpha U^\alpha)\,P + U^\alpha\nabla_\alpha P = \frac{1}{2}\,\gamma^{\alpha\beta}\nabla_\alpha\partial_\beta P$$

or

$$(\nabla_\alpha U^\alpha)P + U^\alpha\nabla_\alpha P = -\,\nabla_\alpha\gamma^{\alpha\beta}\left(P\tilde{b}_\beta - \frac{1}{2}\,\partial_\beta P\right) \tag{I.27}$$

which can be written as

$$(\nabla_\alpha U^\alpha) P + \nabla_U P = -\operatorname{div}_\gamma \gamma\left(\left(b^\# - \frac{1}{2}\nabla\right) P\right) \tag{I.28}$$

where $\operatorname{div}_\gamma$ denotes the divergence operator induced by the Newtonian connection ∇ and $b^\#$ is the one-form locally defined by:

$$b^\# = \tilde{b}_\alpha \, dx^\alpha . \tag{I.29}$$

It appears that (I.28) is the covariant expression on a curved Newtonian space-time of the Fokker-Planck equation. Indeed, it can be verified that the usual form of the Fokker-Planck equation (I.13) is recovered by expressing (I.28) in the special adapted coordinate system where:

$$\gamma = \begin{pmatrix} 1\!\!1_n & \\ & 0 \end{pmatrix}, \quad \psi = (0, 1), \qquad U = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{I.30}$$

Note that the only quantity possessing a physical meaning, that is the Newtonian drift viewed as a vector field, is recovered as

$$b = \gamma(b^\#) \tag{I.31}$$

So, *by introducing an extended evolution space, it has been possible to give an intrinsic formulation of the Fokker-Planck equation through the differential system (I.15 - 16),* and we will call Fokker-Planck operator the differential operator F defined by

$$F(\,.\,) = \operatorname{div}_{\tilde{g}}\left(\tilde{g}^{-1}\left(\left(\tilde{b} - \frac{1}{2}\nabla_{\tilde{g}}\right) \cdot \right)\right). \tag{I.32}$$

Note also that the probability density function P can be replaced by the density:

$$P^{\natural} = |\nu| P \tag{I.33}$$

where $|\nu|$ denotes the Galilean density of (V_{n+1}, γ, ψ).

Then, since

$$L_U \nu = (\nabla_\alpha U^\alpha)\, \nu \tag{I.34}$$

the covariant expression (I.28) of the Fokker-Planck equation can be written as:

$$L_U P^{\natural} = -\operatorname{div}_\gamma\left(\left(b^\# - \frac{1}{2}\nabla\right) P^{\natural}\right). \tag{I.35}$$

To end this section it is important to stress upon the fact that *the metric $\tilde{g}$ carries in itself an external field V which appears in the local expressions (I.3)* *and (I.5) explicitely. But this external field does not appear explicitely anymore in the covariant but local expressions of the Fokker-Planck equation given by (I.28) and (I.34). On a particular example described in Sect. (II.B) it will be shown that it is through the drift term that the external field must reveal its existence.*

II.A - THE CHRONOPROJECTIVE GEOMETRY

It is not the place here to give a complete description of what has been called the (extended) chronoprojective geometry and which has been described in [2]. We want only to briefly recall the properties we need for the following by keeping the point of view we chose to adopt in [2], that is the description of the chronoprojective geometry of $(V_{n+1,1}; \tilde{g}; \xi)$ as a «reduction» of the conformal geometry of $(V_{n+1,1}; \tilde{g})$ governed by the presence of the privileged vector field ξ.

By conformal geometry of a Lorenzian manifold we mean the conformal Cartan structure $(CE(V_{n+1,1}), \omega_{\text{conf}})$ where $CE(V_{n+1,1})$ denotes the reduction of the second-order frames bundle $P^2(V_{n+1,1})$ over $V_{n+1,1}$ to the principal fibre bundle with structure group the conformal Euclidian group of $V_{n+1,1}$ (i.e. the semi-direct product of a dilation by the Euclidian group of $V_{n+1,1}$) endowed with the canonical Cartan connection for $CE(V_{n+1,1})$ with values into the Lie algebra $o(n+2,2)$.

The conformal Cartan structure $(CE(V_{n+1,1}), \omega_{\text{conf}})$ allows to deal with all the families of conformally equivalent Lorentzian metrics over $V_{n+1,1}$.

Now let us introduce the conformal Lorentzian structures $(CO(V_{n+1,1}), \tilde{\varphi}_{\text{conf}})$ where $CO(V_{n+1,1})$ denotes the reduction of the first order frames bundle $P^1(V_{n+1,1})$ over $V_{n+1,1}$ to the principal fibre bundle with structure group the conformal orthogonal group of $V_{n+1,1}$ that is $CO(n+1,1)$ and $\tilde{\varphi}_{\text{conf}}$ is the affine $ce(n+1,1)$-valued connection i.e. $\tilde{\varphi}_{\text{conf}} = (\Theta, \tilde{\Phi}_{\text{conf}})$ where $\Theta = \{\theta^a\}$ ($a \in [1, n+2]$, anholonomic indices) denotes the restriction of the $\mathbb{R}^{n+2}$-valued canonical form of $P^1(V_{n+1,1})$ to $CO(V_{n+1,1})$ and $\tilde{\Phi}_{\text{conf}}$ a $co(n+1,1)$-valued reduction of a torsionless linear connection.

Let us consider an embedding $i : CO(V_{n+1,1}) \to CE(V_{n+1,1})$ and define $\tilde{\varphi}_{\text{conf}} = i^*\omega_{\text{conf}}$. Then it is said that the conformal Lorentzian structure $(CO(V_{n+1,1}), \tilde{\varphi}_{\text{conf}})$ belongs or is subordinate to the conformal Cartan structure $(CE(V_{n+1,1}), \omega_{\text{conf}})$.

Then we can compare the connection one-forms of two conformal Lorentzian structures $(CO(V_{n+1,1}), \tilde{\varphi}_{\text{conf}})$ and $(CO(V_{n+1,1}), \tilde{\varphi}'_{\text{conf}})$ belonging to the same conformal Cartan structure. The expression has been given in [2] and can be locally expressed on the corresponding Christoffel's symbols in a normal local coordinate system $\{x^\mu, \mu \in [1, n+2]\}$ of $V_{n+1,1}$ as:

$$\Gamma'^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\mu\nu} = \delta^{\lambda}_{\mu}\eta_\nu + \delta^{\lambda}_{\nu}\eta_\mu - g_{\mu\nu}g^{\lambda\rho}\theta_\rho \tag{II.1}$$

where $\{\eta_\mu\}$ are the components of an arbitrary one-form η on $V_{n+1,1}$.

To treat the problem of one family of conformally equivalent metrics on V in the sense

$$g' \sim g \quad \text{if} \quad g' = \rho\, g \tag{II.2}$$

where ρ is a positive suitably differentiable function on V, one needs to consider only one conformal Lorentzian structure. To compare the Levi-Cività connections corresponding to two conformally related metrics we have to consider the embeddings of the two corresponding orthogonal frames bundle $O(V)$ into the Lorentzian conformal structure. Then Rel. (II.1) is recovered but now the one-form η is precisely given by

$$\eta = \mathrm{d}\,(\mathrm{Log}\, \rho^{1/2}). \tag{II.3}$$

As recalled above, the chronoprojective geometry can be viewed as a «reduction» of the conformal one, taking into account besides the conformal equivalence of the metrics, the conformal equivalence of the privileged null vector fields defined as

$$\xi' \sim \xi \quad \text{if} \quad \xi' = \sigma\, \xi \tag{II.4}$$

where σ is a priori any function on $V_{n+1,1}$.

But from the parallel transport of $\tilde{g}$ and ξ one deduces that

– σ is a constant function on $V_{n+1,1}$

– ρ is a constant function on the fibres $V_{n+1,1} \to V_{n+1}$, and more, is the pull-back of a function of the «time axis» of V_{n+1} only.

The second order Cartan structure $(L^0(V), \omega_{\mathrm{chr}})$ which is used to describe the chronoprojective geometry is a reduction of $CE(V_{n+1,1})$ to a bundle $L^0(V_{n+1,1})$ where

$$L^0 = \mathbb{R}^n (\times (O(n) \otimes \mathbb{R} \otimes S_2) = H_n (\times (\mathbb{R} \otimes S_2) \subset CE(n+1, 1) \tag{II.5}$$

where H_n denotes the homogeneous Galilei group defined in Sect. I.A and S_2 denotes the two-dimensional solvable group, endowed with the corresponding reduction of ω_{conf} to what we call the chronoprojective connection ω_{chr} which takes its values into the chronoprojective Lie algebra:

$$\widetilde{chr}_n = w_n \,\square\, (o(n) \oplus gl(2, \mathbb{R})) \tag{II.6}$$

where w_n denotes the Lie algebra of the Weyl group (also called the Heisenberg algebra) of dimension $2n+1$, which is a subalgebra of $o(n+2,2)$.

At the first-order frames level, the so-called extended conformal Galilean structures $(L^I(V), \tilde{\varphi})$ have been introduced from the conformal Lorentzian ones $(CO(V_{n+1,1}), \tilde{\varphi}_{\text{conf}})$ by reduction of the bundle $CO(V_{n+1,1})$ to the subbundle $L^I(V_{n+1,1})$ whose structure group L^I is the stabilizer into $CO(n+1, 1)$ of a null direction of $\mathbb{R}^{n+1,1}$. In fact L^I consists into the semi-direct product of the homogeneous Galilei group by two dilations denoted $\dot{\mathbb{R}}_s$ and $\dot{\mathbb{R}}_t$:

(II.7) $$L^I \approx H_n (\times (\dot{\mathbb{R}}_s \otimes \dot{\mathbb{R}}_t).$$

The extended conformal Galilean connection $\tilde{\varphi}$ can be written as $\{\theta, \tilde{\Phi}\}$ where θ denotes here the reduction to $L^I(V_{n+1,1})$ of the canonical form of $P^I(V_{n+1,1})$ and $\tilde{\Phi}$ the l^I-valued reduction of a linear connection. The one-forms $\tilde{\varphi}$ and $\tilde{\varphi}'$ of two extended conformal Galilean connections belonging to the same chronoprojective structure $(L^0(V_{n+1,1}), \omega_{\text{chr}})$ can be compared and locally the corresponding Christoffel's symbols are always related according to (II.1) but here η is proportional to $g(\xi)$.

If we restrict ourselves to a Galilean structure $(H(V), \varphi)$ reduction of a Lorentzian one, the Levi-Cività connection gives rise to a Newtonian connection Φ and $\varphi = (\theta, \Phi)$. If we compare two Galilean structures $(H(V), \varphi)$ and $(H(V), \varphi')$ belonging to the same conformal Galilean structure $(L^I(V), \tilde{\varphi})$ one gets always Rel. (II.1) with the constraint (II.3): $\eta = \mathrm{d}(\mathrm{Log}\, \rho^{1/2})$, but remember that ρ is the pull-back of a function of the «time-axis» of V_{n+1} only.

From Rel. (II.1) the following relations expressing the chronoprojective equivalence on the covariant derivative of a one-form α and on the Laplace operator $\Delta_{\tilde{g}}$ over functions are easily deduced

(II.8) $$\nabla_{\tilde{g}'} \alpha = \nabla_{\tilde{g}} \alpha + n/2\, \rho^{-1} \tilde{g}^{-1}(\mathrm{d}\rho, \alpha)$$

(II.9) $$\Delta_{\tilde{g}'} f = \rho^{-1} \Delta_{\tilde{g}} f + n/2\, \rho^{-2} \tilde{g}^{-1}(\mathrm{d}\rho, \mathrm{d}f).$$

II.B - THE STRUCTURAL INVARIANCE OF THE FOKKER-PLANCK EQUATION

Now we want to study the behaviour of the Fokker-Planck system with respect to the above described geometric structure. This amounts to ask the following question: if $\tilde{P}$ is a solution of a Fokker-Planck system, is it possible to construct $\tilde{P}'$ which will be a solution of the system deduced from the previous one by chronoprojective equivalence.

Let us set

(II.10) $$\tilde{P}' = \rho^{-n/2} \tilde{P}$$

(II.11) $$\tilde{b}' = \tilde{b} - (n/4)\, \mathrm{d}\rho/\rho.$$

The left hand side of (I.21) can be calculated by using (II.8), which leads to

$$\tilde{\nabla}_{\tilde{g}'}(\tilde{g}'^{-1}(\tilde{P}'\tilde{b}')) =$$

(II.12) $$= \rho^{-(n+2)/2}\nabla_{\tilde{g}}\tilde{g}^{-1}(\tilde{P}\tilde{b}) - (n/4)\rho^{-(n+4)/2}\tilde{g}^{-1}(\mathrm{d}\tilde{P}, \mathrm{d}\rho) + + (n^2/16)\rho^{-(n+2)/2}\tilde{P}\tilde{g}^{-1}(\mathrm{d}\rho, \mathrm{d}\rho)$$

and for the right side, by using (II.9) one gets

(II.13) $$\Delta_{\tilde{g}'}\tilde{P}' = \rho^{-(n+2)/2}\Delta_{\tilde{g}}\tilde{P} - - (n/2)\,\rho^{-n/2}\tilde{g}^{-1}(\mathrm{d}\rho, \mathrm{d}\tilde{P}) + (n^2/8)\,\rho^{-(n+2)/2}\tilde{P}\tilde{g}^{-1}(\mathrm{d}\rho, \mathrm{d}\rho).$$

So the following identity is established

(II.14) $$\mathrm{div}_{\tilde{g}'}(\tilde{g}'^{-1}((\tilde{b}' - 1/2\,\nabla_{\tilde{g}'})\tilde{P}') = \rho^{-(n+2)/2}\mathrm{div}_{\tilde{g}}(\tilde{g}^{-1}(\tilde{b} - 1/2\,\nabla_{\tilde{g}})\tilde{P}).$$

In what concerns (I.16), the chronoprojectively transformed quantities automatically satisfy $\xi'(\tilde{P}') = 0$ owing to $\xi(\tilde{P}) = 0$ but the condition (I.17) on the drift having to be always satisfied, that is $\xi' \lrcorner b' = 1$ it implies to take $\sigma = 1$. Therefore the identity (II.14) and the above considerations show that *if $\tilde{P}$ is a solution of a Fokker-Planck system, $\tilde{P}'$ is also a solution of the chronoprojectively equivalent one i.e. the dimension of the space of Fokker-Planck system solutions is a chronoprojective invariant*. This is what is meant when we speak of *structural invariance of Fokker-Planck systems under the chronoprojective geometry*.

Now, let us show how this invariance property can be used to connect various systems described by the Fokker-Planck formalism.

Let us consider the chronoprojectively flat extended configuration space-time $(\mathbb{R}^{n+2}, \tilde{g}_0, \xi = \partial_{y^{n+2}})$ with

(II.15) $$\tilde{g}_0 = \delta_{jk}\,\mathrm{d}y^j \otimes \mathrm{d}y^k + \mathrm{d}y^{n+1} \otimes \mathrm{d}y^{n+2}$$

Let $\tilde{A}$ be the local diffeomorphism of $\mathbb{R}^{n+2}$ $\tilde{A} : (\mathbf{x}, t, s) \to (\mathbf{y}, y^{n+1}, y^{n+2})$ given by:

(II.16) $$\begin{cases} \mathbf{y} = (\mathbf{x} + \boldsymbol{\beta})r^{-1} \\ y^{n+1} - y_0^{n+1} = T(\tau) = \int r(\tau)^{-2}\,\mathrm{d}\tau, \qquad \tau = t - t_0 \\ y^{n+2} = s + 1/2\,(\mathrm{Log}\,r)(\mathbf{x} + \boldsymbol{\beta})^2 - (\dot{\boldsymbol{\beta}}.\mathbf{x}) + 1/2\,\gamma \end{cases}$$

where r is a solution of $\ddot{r} = \alpha^2 r$ such that $r(o) = 1$; r, α^2, β^j, γ being real functions of the time coordinate t only, and the dot denoting the derivation with respect to t.

Under the diffeomorphism $\tilde{A}$, $\mathbb{R}^{n+2}$ is endowed with a new chronoprojectively flat metric $\tilde{g}_V$ defined by

(II.17) $$\tilde{A} * \tilde{g}_0 = r^{-2} \tilde{g}_V$$

and which can be expressed as

(II.18) $$\tilde{g}_V = \delta_{jk}\, dx^j \otimes dx^k + ds \otimes dt - 2V\, dt \otimes dt$$

with

(II.19) $$V = -1/2\, \{\alpha^2 \mathbf{x}^2 + 2((\alpha^2 \boldsymbol{\beta} - \ddot{\boldsymbol{\beta}}) \,.\, \mathbf{x}) + \alpha^2 \boldsymbol{\beta}^2 + \dot{\boldsymbol{\beta}}^2 + \dot{\gamma}\}.$$

This potential V expresses the inhomogeneity of the ambient medium. To study the $(n+2)$-dimensional Fokker-Planck equation written for the metric $\tilde{g}_0$ turns out to treat the following equation in the $(n+1)$-dimensional «physical space»

(II.20) $$\partial_{y^{n+1}} P = - \sum_{j \in [1,n]} \left\{ \partial_{y^j}(bP) - \frac{1}{2}\, \partial_{y^j}\, \partial_{y^j} P \right\}$$

where the coordinate y^{n+1} plays the role of the time.

If a solution P of this equation is known for a particular drift b, owing to the chronoprojective invariance which exists at the level of the extended space-time, a solution P' of the Fokker-Planck equation

(II.21) $$\partial_t P' = - \sum_j \left\{ \partial_{x^j}(b'P') - \frac{1}{2}\, \partial_{x^j}\, \partial_{x^j} P' \right\}$$

can be deduced by

(II.22) $$P'(\mathbf{x}, t) = r(t)^{-n} (P \circ A)(\mathbf{x}, t)$$

where A denotes the restriction of $\tilde{A}$ to the $(n+1)$ first coordinates i.e. $A : (\mathbf{x}, t) \rightarrow (\mathbf{y}, y^{n+1})$ and corresponds to the drift

(II.23) $$b^{\#\prime}(\mathbf{x}, t) = (b^{\#} \circ A)(\mathbf{x}, t) - n(\dot{\mathrm{Log}}\, r)/2 + d\left\{ (\dot{\mathrm{Log}}\, r)\, \frac{1}{2}\, (\mathbf{x} + \boldsymbol{\beta})^2 - (\dot{\boldsymbol{\beta}} \,.\, \mathbf{x}) + \gamma/2 \right\}.$$

It is interesting to remark that *the inhomogeneity of the medium expressed by the presence of the potential V into the expression of the metric $\tilde{g}_V$ manifests implicitely only through the modified drift b' and not by the emergence of a supplementary multiplicative term into the Fokker-Planck operator*, which corroborates the final remark of Sec. I.B.

Let us give now the example of a continuous distribution that appears very frequently namely the «normal» (or Gaussian) solution of the Fokker-Planck equation.

Let P be the Gaussian distribution written as

(II.24) $$P = (2\pi\Sigma^2)^{-n/2} \exp(-\mathbf{u}^2/2)$$

where

(II.25) $$\mathbf{u} = \Sigma^{-1}(\mathbf{y} - \mathbf{y}_M)$$

with

(II.26) $$\mathbf{y}_M = \mathbf{y}_0 + \mathbf{v}_0(y^{n+1} - y_0^{n+1})$$

i.e. the average $\mathbf{y}_M$ is the trajectory in $\mathbb{R}^{n+1}$ of a free particle located in $\mathbf{y}_0$ at the date y_0^{n+1} with the velocity $\mathbf{v}_0$; such a distribution is solution of the Fokker--Planck equation for the drift

(II.27) $$\mathbf{b} = (\dot{\Sigma} - 1/2\,\Sigma)\,\mathbf{u} + \dot{\mathbf{y}}_M$$

the variance Σ^2 being the solution of the equation

(II.28) $$\ddot{\Sigma} = 1/(4\,\Sigma^3)$$

(the dot denoting the derivative with respect to the time coordinate y^{n+1}) given by

(II.29) $$\Sigma = [a + b(y^{n+1} - y_0^{n+1}) + c(y^{n+1} - y_0^{n+1})^2]^{1/2}$$

with a and $c \in \mathbb{R}^+$ and $b \in \mathbb{R}$ such that $4ac - b^2 = 1$.

In the $(n+1)$-dimensional covariant formalism, it turns out to consider the drift form

(II.30) $$\begin{aligned} b^{\#} = {}^t((\dot{\Sigma} - 1/(2\Sigma))\mathbf{u} + \mathbf{y}_M)\,.\,\mathrm{d}\mathbf{y} + \\ + \{(1/(4\,\Sigma^2) + \dot{\Sigma}/\Sigma - \dot{\Sigma}^2)\,\mathbf{u}^2 + (1/\Sigma - 2\dot{\Sigma})(\mathbf{u}\,.\,\mathbf{y}_M) - \\ - n(\dot{\Sigma} + 1/(2\,\Sigma))/\Sigma - \dot{\mathbf{y}}_M^2\}\,\mathrm{d}y^{n+1}/2. \end{aligned}$$

Through (II.23) one constructs $\mathbf{b}'^{\#}$ from which the following drift vector $\mathbf{b}'$ is deduced

(II.31) $$\mathbf{b}' = (\dot{\Sigma}' - 1/(2\,\Sigma'))\,\mathbf{u}' + \dot{\mathbf{x}}_M$$

where

(II.32) $$\mathbf{u}' = \Sigma'^{-1}(\mathbf{x} - \mathbf{x}_M)$$

with

(II.33) $$\mathbf{x}_M = \mathbf{x}_0 r + \mathbf{v}_0 q - \boldsymbol{\beta}$$

Σ' being now a solution of the Pinney's equation

(II.34) $$\ddot{\Sigma}' - \alpha^2 \Sigma' = 1/(4\, \Sigma'^3)$$

given by

(II.35) $$\Sigma' = [a r^2 + b\, r\, q + c q^2]^{1/2}$$

where $q(\tau) = r(\tau) T(\tau)$ is a solution of the equation $\ddot{q} = \alpha^2 q$ such that $q(o) = o$; then r and q are two independent solutions of the same differential equation since their Wronskian $r\dot{q} - q\dot{r}$ is equal to 1 everywhere.

So, we are now faced with a solution of the equation (II.21) which is still a Gaussian one

(II.36) $$P' = (2\pi \Sigma'^2)^{-n/2} \exp(-\mathbf{u}'^2/2)$$

whose centre $\mathbf{x} = \mathbf{x}_M$ corresponds here to the trajectory of a test principle submitted to the potential V given by (II.19) and whose variance reflects also the presence of V, although a term like VP' does not appear into Eq. (II.21): consequently all the informations concerning the external medium are contained into the expression of the drift $\mathbf{b}'$.

II.C - INVARIANCE ALGEBRA OF THE FOKKER-PLANCK SYSTEM

The symmetry properties of a differential system just reflect the infinitesimal aspect of its structural invariance properties. Hence, according to the properties described in the above Sect. II.B, for the Fokker-Planck system we have to study the automorphisms of the chronoprojective structure described in Sect. II.A.

By definition the automorphisms Aut $(L^0(V_{n+1,1}), \omega_{\mathrm{chr}})$ of an extended chronoprojective structure are the elements of Aut $(L^0(V_{n+1,1}))$ induced by the diffeomorphisms of $V_{n+1,1}$ which map the extended chronoprojective connection onto itself. Owing to the existence of a Cartan connection, $L^0(V_{n+1,1})$ is parallelizable and consequently Aut $(L^0(V_{n+1,1}), \omega_{\mathrm{chr}})$ is a finite-dimensional Lie group such that

(II.37) $$\dim \mathrm{Aut}\, (L^0(V_{n+1,1}), \omega_{\mathrm{chr}}) \leqslant \dim L^0(V_{n+1,1}) = (1/2) n(n+3) + 5.$$

These automorphisms are in one-to-one correspondence with the automorphisms of $L^I(V_{n+1,1})$ which map an admissible connection onto a chronoprojectively equivalent one.

On $V_{n+1,1}$ they correspond to the elements of Diff $(V_{n+1,1})$ which ensure the chronoprojective equivalence, namely $g' = \rho g$ and $\xi' = \sigma \xi$ where ρ and σ are the functions described in Sect. II.A. At each stage such mappings will be called chronoprojective transformations.

At the infinitesimal level every vector field X on $V_{n+1,1}$ generates a one-parameter (local) group of (local) transformations. This local group of transformations prolonged to the bundle of first-order frames $P^1(V_{n+1,1})$ and to the bundle of second-order frames $P^2(V_{n+1,1})$ induces a vector field $\bar{X}$ on $P^1(V_{n+1,1})$ and a vector field $\bar{\bar{X}}$ on $P^2(V_{n+1,1})$. Then a vector field X will be called an infinitesimal extended chronoprojective transformation ($\in$ aut $(L^0(V_{n+1,1}), \omega_{\mathrm{chr}})$) if the local one-parameter group generated by X in a neighbourhood of each point of $V_{n+1,1}$ consists of local extended chronoprojective transformations.

Let us give explicitely the equations which lead to the realizations of the Lie algebra of automorphisms on the bundle of second-order frames and on the basis.

i) on $P^2(V_{n+1,1})$

(II.38) $$L_{\bar{\bar{X}}}\,\omega_{\mathrm{chr}} = 0$$

ii) on $V_{n+1,1}$

(II.39) $$L_X g = \epsilon g, \quad L_X \xi = \kappa \xi.$$

Now let us show how the structural invariance properties of the Fokker-Planck system leads to its symmetry algebra. First of all note that the structural invariance is obtained by restricting the chronoprojective equivalence to $\xi' = \xi$ (i.e. corresponding to $\sigma = 1$). This implies to have to work with the Lie algebra aut $(L^0(V_{n+1,1}), \omega_{\mathrm{chr}})$ restricted by the condition $\kappa = 0$, that is to consider the chronoprojective vector fields which keep invariant the ξ fibration. In the case where the maximal dimension of the automorphisms group is reached (which corresponds to what is called a chronoprojectively flat structure) the symmetry algebra is known as the (extended) Schrödinger algebra $\widetilde{sch}_n \approx w_n \square\, (o(n) \oplus sl(2, \mathbb{R}))$, otherwise it is a subalgebra of $\widetilde{sch}_n$ which is obtained only.

Let us supposed that

(II.40) $$L_X \tilde{P} = k\epsilon\, \tilde{P}$$

and look for determining the constant k. By a direct calculation

(II.41) $$L_X(\Delta_{\tilde{g}}\tilde{P}) = (k-1)\epsilon\, \Delta_{\tilde{g}}\tilde{P} + g^{-1}(\mathrm{d}\,(2k + n/2)\boldsymbol{\epsilon}, \mathrm{d}\tilde{P}) + k\tilde{P}\,\Delta_{\tilde{g}}\epsilon.$$

The vanishing of the curvature $\tilde{R}$ implies $\Delta_{\tilde{g}}\epsilon = 0$ since

(II.42) $$L_X \tilde{R} = (n+1)\,\Delta_{\tilde{g}}\epsilon - \epsilon\tilde{R}.$$

We are then left with $L_X(\Delta_{\tilde{g}}\tilde{P}) = (k-1)\epsilon\, \Delta_{\tilde{g}}\tilde{P} + \tilde{g}^{-1}(\mathrm{d}\,(2k + n/2)\epsilon, \mathrm{d}\tilde{P})$ which is nothing else than the infinitesimal form of Rel (II.13).

In the same manner we establish that

$$L_X \nabla_{\tilde{g}}(\tilde{g}^{-1}\tilde{b}\tilde{P}) = \epsilon(k-1)\Delta_{\tilde{g}}(\tilde{b}\tilde{P}) + + (k+n/2)\tilde{g}^{-1}(\mathrm{d}\epsilon, \tilde{b}\tilde{P}) + n\tilde{g}^{-1}(\mathrm{d}\epsilon, \mathrm{d}\tilde{P})/4 \tag{II.43}$$

by using the infinitesimal form of (II.11), namely

$$L_X \tilde{b} = n\,\mathrm{d}\epsilon/4. \tag{II.44}$$

Finally one obtains

$$L_X(1/2\,\Delta_{\tilde{g}}\tilde{P} - \nabla_{\tilde{g}}(\tilde{g}^{-1}(\tilde{b}\tilde{P}))) = -\epsilon(n+2)(1/2\,\Delta_{\tilde{g}}\tilde{P} - \nabla_{\tilde{g}}(\tilde{g}^{-1}(\tilde{b}\tilde{P})))/2 \tag{II.45}$$

by setting $k = -n/2$ which confirms the cotransformation property of $\tilde{P}$ given by (II.10): it is in fact the infinitesimal form of (II.14) which can be expressed by

$$L_X F = -\epsilon F - F(\epsilon) = -F\epsilon \tag{II.46}$$

where F is the Fokker-Planck operator defined in Sect. I.B.

This result can also be expressed as a Lie bracket of differential operators under the following form

$$[X - k\epsilon, F] = -\epsilon F. \tag{II.47}$$

Then, *if we forget the geometrical background the invariance algebra of the Fokker-Planck operator can be directly found by solving the system of differential equations which derives from the above relation (II.47): the generalization to manifolds of the Lie's trick.*

In the particular case of a linear drift such a calculation has been performed [5] for $n = 1$ and $n = 2$ and has really led to the Schrödinger algebras $\widetilde{sch}_1$ and $\widetilde{sch}_2$.

III - CONCLUDING REMARKS

In the last past years numerous works have been performed about the relation between the Fokker-Planck equation and the Schrödinger one. This have led, among other things, to what is called now the stochastic formulation of quantum mechanics. So it can be asked whether this correlation remains in the above described geometric formalism on an extended space-time.

The geometric formulation of the Schrödinger equation has been described in [3] where it appears as the following set of partial differential equations:

$$\Delta_{\tilde{g}}\tilde{\psi} = 0 \tag{III.1}$$

$$\xi\,\tilde{\psi} = (\mathrm{i}m/\hbar)\,\tilde{\psi} \tag{III.2}$$

showing that the quantum states are in fact the harmonic functions on $V_{n+1,1}$ satisfying a particular property with respect to the character of $(\mathbb{R}, +)$.

If one sets as per usual:

(III.3) $$\tilde{\psi} = \tilde{R} \exp (i \tilde{S})$$

the system (III.1 - 2) becomes

(III.4a) $$2\, \tilde{g}^{-1}(\tilde{R}^{-1} \mathrm{d}\tilde{R}, \mathrm{d}\tilde{S}) + \Delta_{\tilde{g}} \tilde{S} = 0$$

(III.4b) $$\tilde{R}^{-1} \Delta_{\tilde{g}} \tilde{R} - \tilde{g}^{-1}(\mathrm{d}\tilde{S}, \mathrm{d}\tilde{S}) = 0$$

(III.5a) $$\xi(\tilde{R}) = 0$$

(III.5b) $$\xi(\tilde{S}) = 1.$$

Eq. (III.4a) which can be written

(III.6) $$\mathrm{div}_{\tilde{g}}(\tilde{g}^{-1}(\tilde{R}^2 \mathrm{d}\tilde{S})) = 0$$

corresponds in fact to the conservation equation for the Schrödinger current. Then, by setting

(III.7) $$\tilde{b} = \tilde{R}^{-1} \mathrm{d}\tilde{R} + \mathrm{d}\tilde{S} = (\tilde{R}^{-2} \mathrm{d}(\tilde{R}^2))/2 + \mathrm{d}\tilde{S}$$

and by identifying $\tilde{R}^2$ with $\tilde{P}$, the current conservation equation together with (III.5) corresponds to the Fokker-Planck system (I.15 - 16). So *the correlation between Schrödinger systems and Fokker-Planck ones is maintained in our formalism*, but the stochastic quantization aspect cannot be entered upon in the existing state of our approach. Let us recall that the Fokker-Planck equation is an evolution equation for the probability distribution of fluctuating macroscopic variables linked to a stochastic description of a physical system endowed with a microscopic structure whose description is discarded. All the informations are then contained into the differential equation relating the macroscopic variables to a chosen stochastic process. This aspect remains to be developed in the above described formalism.

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